Corrigendum to "Computation of free surface waves in coastal waters with SWASH on unstructured grids" [Comput. Fluids 213 (2020)]

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Unfortunately, the original version of the article contained errors in the presentation of the proof of momentum conservation (see Appendix A of the original article). The correct and final version of the proof is presented in this note. Note that the meaning of the variables, parameters and indices in the equations below can be found in the original article.

The objective of this note is to demonstrate that the semi-discretized momentum equation (see the original article for the construction of the discretization), for $u_f > 0$,

$$\frac{du_f}{dt} + \frac{\Delta s_{c_\mathrm{L},f}}{\Delta s_f \, \tilde{h}_f \, A_{c_\mathrm{L}}} \left[Q_{k_1} \left(u_f - \mathbf{u}_{c_{\mathrm{L},k_1}} \cdot \mathbf{n}_f \right) + Q_{k_2} \left(u_f - \mathbf{u}_{c_{\mathrm{L},k_2}} \cdot \mathbf{n}_f \right) \right] + g \, \frac{\overline{h}_f}{\tilde{h}_f} \, \frac{\zeta_{c_\mathrm{R}} - \zeta_{c_\mathrm{L}}}{\Delta s_f} = -\frac{1}{2\tilde{h}_f} \frac{h_{c_\mathrm{R}} p_{c_\mathrm{R}} - h_{c_\mathrm{L}} p_{c_\mathrm{L}}}{\Delta s_f} + \frac{\tilde{p}_f}{\tilde{h}_f} \frac{d_{c_\mathrm{R}} - d_{c_\mathrm{L}}}{\Delta s_f} - c_f \frac{u_f \|\tilde{\mathbf{u}}_f\|}{\tilde{h}_f} \tag{1}$$

as implemented in UnSWASH, does not produce a momentum conservation error, given a uniform bed and a zero bed friction. By means of

$$g \, \overline{h}_f \left(\zeta_{c_\mathrm{R}} - \zeta_{c_\mathrm{L}} \right) = \frac{1}{2} \, g \left(h_{c_\mathrm{R}}^2 - h_{c_\mathrm{L}}^2 \right) \tag{2}$$

we obtain the following finite difference form

$$\Delta s_f \frac{du_f}{dt} + \frac{\Delta s_{c_\mathrm{L},f}}{A_{c_\mathrm{L}}} \left[\frac{Q_{k_1}}{\tilde{h}_f} \left(\mathbf{u}_f - \mathbf{u}_{c_{\mathrm{L},k_1}} \right) + \frac{Q_{k_2}}{\tilde{h}_f} \left(\mathbf{u}_f - \mathbf{u}_{c_{\mathrm{L},k_2}} \right) \right] \cdot \mathbf{n}_f = -\frac{1}{\tilde{h}_f} \left(P_{c_\mathrm{R}} - P_{c_\mathrm{L}} \right) \tag{3}$$

where

$$P_c = \frac{1}{2} \left(g h_c^2 + h_c p_c \right)$$

is the depth-integrated total pressure at the circumcentre. Discrete momentum conservation can be expressed as the rate of change in the total amount of momentum $h\mathbf{u}$ within a control volume is due only to the net flux through the edges of the volume. Since the normal face velocity u_f is the primary unknown, interpolation must be employed to obtain the momentum at a single location within the mesh. To this end, the mesh cell is treated as the control volume. Subsequently, we derive an equation for the cell-based momentum vector using Perot's interpolation scheme [1], as given by

$$\mathbf{u}_c = \frac{1}{A_c} \sum_{f \in S_c} l_f \Delta s_{c,f} \, u_f \, \mathbf{n}_f \tag{4}$$

Finally, summation of its flux contributions over the cell faces must lead to a discrete equivalent of the momentum equation in divergence form, which completes the proof. This formal procedure of proof utilizes the geometric properties of the mesh and is also applicable to Cartesian staggered schemes (see, e.g. [1, 2]).

First, Eq. (3) is rewritten as

$$\tilde{h}_f\,\Delta s_f \frac{du_f}{dt} + \left(\Delta s_{c_{\mathrm{L}},f}\,\mathbf{c}_{c_{\mathrm{L}}} + \Delta s_{c_{\mathrm{R}},f}\,\mathbf{c}_{c_{\mathrm{R}}}\right)\cdot \mathbf{n}_f = -\left(P_{c_{\mathrm{R}}} - P_{c_{\mathrm{L}}}\right) \tag{5}$$

with

$$\mathbf{c}_c = \frac{1}{A_c}\sum_{k\in S_c} \alpha_{c,k}\,Q_k\,\left(\hat{\mathbf{u}}_k - \mathbf{u}_f\right)$$

the cell-based discretization of the advection term evaluated for the two cells adjacent to face f. Note that the transported velocity $\hat{\mathbf{u}}_k$ is interpolated from face normal components from the cell upwind of face k. As a consequence, $\mathbf{c}_{c_{\mathrm{R}}}\cdot\mathbf{n}_f = 0$ if $u_f > 0$, and likewise, $\mathbf{c}_{c_{\mathrm{L}}}\cdot\mathbf{n}_f = 0$ if $u_f < 0$. Using the following expression

$$\mathbf{a}_c = \frac{1}{A_c}\sum_{k\in S_c} \alpha_{c,k} Q_k\,\hat{\mathbf{u}}_k \tag{6}$$

and the continuity equation

$$A_c\frac{dh_c}{dt} + \sum_{k\in S_c}\alpha_{c,k}Q_k = 0 \tag{7}$$

the obtained vector $\mathbf{c}_c$ is rewritten as

$$\mathbf{c}_c = \mathbf{a}_c + \mathbf{u}_f\frac{dh_c}{dt}$$

Substituting into Eq. (5) and using the following relation

$$\Delta s_f\,\tilde{h}_f = \Delta s_{c_{\mathrm{L}},f} h_{c_{\mathrm{L}}} + \Delta s_{c_{\mathrm{R}},f} h_{c_{\mathrm{R}}} \tag{8}$$

we get

$$\tilde{h}_f\,\Delta s_f\frac{du_f}{dt} + u_f\Delta s_f\frac{d\tilde{h}_f}{dt} + \left(\Delta s_{c_{\mathrm{L}},f}\,\mathbf{a}_{c_{\mathrm{L}}} + \Delta s_{c_{\mathrm{R}},f}\,\mathbf{a}_{c_{\mathrm{R}}}\right)\cdot\mathbf{n}_f = -\left(P_{c_{\mathrm{R}}} - P_{c_{\mathrm{L}}}\right)$$

Next, the discretized equation is multiplied by the normal of the face $\mathbf{n}_f$ and its length l_f, and subsequently summed over the faces of cell c

$$\sum_{f\in S_c}\mathbf{n}_f l_f\left(\Delta s_{c_{\mathrm{L}},f} + \Delta s_{c_{\mathrm{R}},f}\right)\frac{d\tilde{h}_f u_f}{dt} + \sum_{f\in S_c} l_f\left(\Delta s_{c_{\mathrm{L}},f}\,\mathbf{a}_{c_{\mathrm{L}}} + \Delta s_{c_{\mathrm{R}},f}\,\mathbf{a}_{c_{\mathrm{R}}}\right)\cdot\mathbf{n}_f\mathbf{n}_f = -\sum_{f\in S_c}\mathbf{n}_f l_f\left(P_{c_{\mathrm{R}}} - P_{c_{\mathrm{L}}}\right)$$

We now demonstrate conservation of momentum by transforming the above equation into an equation for the cell-based vector $h\mathbf{u}$, i.e. the total amount of momentum per unit cell area. Each interior face is shared by two triangular cells of which one contributes to the cell under consideration. Furthermore, at the boundary face a flux of momentum is prescribed.

First, the rate of change in momentum can be recasted as

$$\sum_{f\in S_c}\mathbf{n}_f l_f\Delta s_{c,f}\frac{d\tilde{h}_f u_f}{dt} = \frac{d}{dt}\sum_{f\in S_c}\mathbf{n}_f l_f\Delta s_{c,f}\tilde{h}_f u_f = A_c\frac{d\left(h\mathbf{u}\right)_c}{dt}$$

following from Eq. (4). This holds for all cells in the computational mesh.

Next, the advective acceleration term can be rearranged with the aid of the following geometric identity

$$\sum_{f \in S_c} l_f \Delta s_{c,f}\, \mathbf{n}_f \mathbf{n}_f = A_c\, \mathbf{I}$$

with $\mathbf{I}$ the identity matrix. This identity follows from the divergence theorem [1]. Then, using Eq. (6), the advection term can be expanded as

$$\sum_{f \in S_c} l_f \Delta s_{c,f}\, \mathbf{a}_c \cdot \mathbf{n}_f \mathbf{n}_f = \mathbf{a}_c \cdot \sum_{f \in S_c} l_f \Delta s_{c,f}\, \mathbf{n}_f \mathbf{n}_f = \mathbf{a}_c \cdot \mathbf{I} A_c = A_c \mathbf{a}_c = \sum_{k \in S_c} \alpha_{c,k}\, Q_k\, \hat{\mathbf{u}}_k$$

which conserves momentum in the considered cell since $\alpha_{c_\mathrm{R},k} = -\alpha_{c_\mathrm{L},k}$ at each interior face k. This implies that the sum of contributions to the left and right cells cancels the advective flux at face k. This holds for all cells except the boundary cells where the change in momentum is due to the fluxes across the boundary faces.

Finally, the pressure term in the interior cell can be rewritten as

$$\sum_{f \in S_c} \mathbf{n}_f l_f P_c = -P_c \sum_{f \in S_c} \mathbf{n}_{c,f}\, l_f = 0$$

by noting that the pressure gradient is aligned with the flow direction and the cell has a closed surface. In case the cell under consideration is a boundary cell, then

$$\sum_{f \in S_c} \mathbf{n}_f l_f P_c = \sum_{f \in S_c} \mathbf{n}_{c,f}\, l_f\, P_f$$

with P_f the contribution to the prescribed momentum flux.

In short, by rewriting the finite difference form (1) into a discrete equation in divergence form without introducing any other approximation, we have shown that it does not create or destroy momentum in each individual mesh cell of the computational domain. This implies that the computed amount of momentum can only change as a result of a non-zero net momentum flux over the boundary of the domain, a non-uniform bed, or a non-zero bed friction.

References

[1] B. Perot, Conservation properties of unstructured staggered mesh schemes, J. Comput. Phys. 159 (2000) 58–89.

[2] Y. Morinishi, T. S. Lund, O. V. Vasilyev, P. Moin, Fully conservative higher order finite difference schemes for incompressible flow, J. Comput. Phys. 143 (1998) 90–124.